Perverse Sheaves

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1 Introduction

Perverse sheaves were introduced by Beilison, Bernstein and Deligne [BBD82] to provide a sheaf theoretic foundation for the theory of intersection cohomology as developed by Goresky-Macpherson [GM80]. This allows (crucially) for the theory to be defined in the étale cohomology of varieties over finite (or characteristic p) fields and this allows one to deploy the general machinery of weights as was developed earlier by Deligne in [Del80].

Intersection cohomology is a way to take cohomology of a singular variety (or pseudomanifold) that agrees with ordinary cohomology for smooth projective varieties (manifolds). It is better behaved than ordinary cohomology because it satisfies a version of Poincaré duality and because it has a naturally defined cup product.

The theory of perverse sheaves has found many applications, for example in representation theory and the geometric Langlands programme. For example Beilison and Bernstein proved the Kahzdan-Lustzig conjecture by studying the intersection cohomology on Schubert varieties.

The most striking application is the decomposition theorem, which generalises (to the nonsmooth case) the following theorem of Deligne:

Theorem 1. *Let $f : X \to Y$ be a smooth proper morphism of varieties over an algebraically closed field k, then (for l coprime to the characteristic of k) we have*

$$Rf_*\mathbb{Q}_l \cong \bigoplus_i R^i f_* \mathbb{Q}_l[-i],$$

which for example implies that the Leray spectral sequence for f degenerates.

2 Six functor formalism

In this section we will introduce, for schemes of finite type over an algebraically closed field k, the bounded derived category of l-adic sheaves with constructible cohomology. Actually, we will only define things with finite (e.g. $\mathbb{Z}/l^n\mathbb{Z}$ coefficients), and will not deal with the problems of passing to the limit. We refer to Section 6 [BS15] for a modern approach, which also deals with things in more generality. We will briefly recall constructible sheaves and their basic properties, as well as the definition of a triangulated category. Next, we define the category $D_c^b(X, \Lambda)$ and discuss functoriality for morphisms $f : X \to Y$ of finite type schemes.

2.1 Constructible Sheaves

In this section X is a scheme of finite type over a field k, there is a coefficient ring $\Lambda = \mathbb{Z}/l^n\mathbb{Z}$ for l coprime to the characteristic of k.

Definition 1. *A sheaf of Λ-modules $\mathcal{F}$ on the étale site of X is constructible if there is a stratification $X = \bigcup_i U_i$ into locally closed subschemes U_i such that $\mathcal{F}\big|_{U_i}$ is finite locally constant. This means that there is an étale cover $V_i \to U_i$ such that $\mathcal{F}\big|_{U_i}$ becomes a constant sheaf after pullback to V_i.*

We let $\mathbf{Ab}_c(X_{\text{ét}}, \Lambda) \subset \mathbf{Ab}(X_{\text{ét}}, \Lambda)$ be the full subcategory of the abelian category of sheaves of Λ-modules on $X_{\text{ét}}$ whose objects are constructible sheaves. We will need the following result [Sta18, Tag 03RY]

Proposition 1. *The subcategory $\mathbf{Ab}_c(X_{ét}, \Lambda) \subset \mathbf{Ab}(X_{ét}, \Lambda)$ is a strong Serre subcategory, i.e., it is closed under sub-quotients and extensions.*

2.2 Triangulated categories

Definition 2. *A triangulated category is a triple $(\mathcal{C}, T : \mathcal{C} \to \mathcal{C}, \Delta)$ where $\mathcal{C}$ is an additive category, $T : \mathcal{C} \to \mathcal{C}$ is an additive functor which is part of equivalence of categories, and Δ is a collection of diagrams (called triangles) of the form*

$$A \to B \to C \to T(A),$$

such that the following axioms hold (we call triangles in Δ distinguished triangles:

T1 *Every triangle that is isomorphic to a distinguished triangle is also a distinguished triangle.*

T2 *For every object X the triangle*

$$X \xrightarrow{id} X \longrightarrow 0 \longrightarrow T(X)$$

is distinguished.

T3 *For every morphism $f : X \to Y$, there is a distinguished triangle (unique up to non-unique isomorphism)*

$$X \to Y \to Z \to T(X)$$

T4 *The triangle*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$$

is distinguished if and only if the following triangle is distinguished

$$Y \xrightarrow{v} Z \xrightarrow{w} T(X) \xrightarrow{-Tu} T(Y).$$

T5 *Given a diagram*

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & T(X) \\ \downarrow f & & \downarrow & & \vdots & & \downarrow Tf \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & T(X') \end{array}$$

with rows distinguished, then the dotted arrow exists (but not uniquely).

T6 *The octahedral axiom.*

A triangulated functor $F : (\mathcal{C}, T, \Delta) \to (\mathcal{C}', \mathcal{T}', \Delta')$ *is an additive functor that sends triangles in* Δ *to* Δ*' such that the following diagram commutes (possibly up to natural transformations??)*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{T} & \mathcal{C} \\ \downarrow F & & \downarrow F \\ \mathcal{C}' & \xrightarrow{T'} & \mathcal{C}'. \end{array}$$

Remark 1. There is a more elegant way of defining all this, using the language of stable infinity categories, see Chapter 1 of [Lur17].

Proposition 2. *If* $\mathbf{A}$ *is a sufficiently nice abelian category (for example the category of sheaves of abelian groups on a site), then the bounded derived category* $D^b(A)$ *is a triangulated category. The objects of this category are bounded chain complexes and morphisms are roughly chain homotopy classes of maps (up to quasi-isomorphisms etc).*

Definition 3. *We define* $D(X) = D^b_c(X, \Lambda)$ *to be the full subcategory of the bounded derived category of* Λ*-modules consisting of complexes* K *whose cohomology sheaves are constructible. Proposition 1 tells us that this is a triangulated subcategory.*

2.3 The four functors

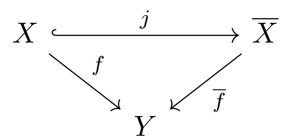
In this section we let $f : X \to Y$ be a finite type morphisms of schemes, then we will consider the various pullback and pushforward functors that f induces between $D(X)$ and $D(Y)$. We know that there are (non-derived) functors $f_* : \mathbf{Ab}(X_{\text{ét}}, \Lambda) \leftrightarrow \mathbf{Ab}(Y_{\text{ét}}, \Lambda) : f^*$. Moreover, the functor f_* is left exact and the functor f^* is exact, and f_* is right adjoint to f^*. The machinery of derived categories gives us functors

$$Rf_* : D^b(X_{\text{ét}}, \Lambda) \to D^b(Y_{\text{ét}}, \Lambda)$$
$$f^* : D^b(Y_{\text{ét}}, \Lambda) \to D^b(X_{\text{ét}}, \Lambda)$$

Proposition 3. *Both functors preserve the subcategories of complexes with constructible cohomology sheaves*

Proof. This is straightforward for f^* and a deep result (Theorem 7.1.1 in [Del77]) for Rf_*, which implies for example finite dimensionality of etale cohomology. □

If the morphism f is separated, then we can define a functor $f_!$ (called pushforward with compact support) which is related to cohomology with compact support. This functor is defined by choosing a proper compactification $\overline{f}$ of the morphism f (which exists, c.f. [Con07])



and defining $f_! K = \overline{f}_* j_! K$ where $j_!$ is the extension by zero functor. This does not depend on the chosen compactification by the proper base change theorem.

Proposition 4. *The functor $f_!$ preserves constructibility*

Proof. This is Théorème 1.1in SGA 4 Exposé XIV ([Gro73]), but it also follows from Deligne's result cited above. □

2.4 Duality

In this section we describe Verdier duality for separated morphisms $f : X \to Y$, which specialises to Poincare duality when Y is a point and f is proper.

Theorem 2 (Théorème 3.1.4 SGA4 Exposé XVIII)**.** *Let $f : X \to Y$ be a finite type separated morphism between finite type schemes over an algebraically closed field, then the functor $f_! : D(X) \to D(Y)$ admits a (triangulated) right adjoint functor $f^!$ called shriek pullback. So this means we have for $K \in D(X), L \in D(Y)$ a natural isomorphism*

$$f_* R\mathscr{H}om(K, f^! L) = R\mathscr{H}om(f_! K, L).$$

Definition 4. *If $f : X \to \operatorname{Spec} k$ is the structure morphism, we define the dualizing complex $D_X = f^! \overline{\mathbb{Q}_l}$ where we put $\overline{\mathbb{Q}_l}$ in degree 0. We define the (contravariant) dualizing functor by*

$$DK = D_X(K) := R\mathscr{H}om(K, K_X).$$

Proposition 5. *The following properties hold:*

- *There is a natural isomorphism $K \to DDK$ (Théorème 4.3 SGA 4.5 [Del77]).*
- *We have $f^! K_Y = K_X$.*
- *We have $f^! \circ D_Y = D_X \circ f^*$.*
- *We have $f_* \circ D_X = D_Y \circ f_!$ (Poincaré duality).*

3 Perverse sheaves

In this section we will define the abelian category of perverse sheaves $\mathrm{Perv}(X)$ for X a finite type scheme over a field k. We will discuss some of its properties and behaviour under morphisms $f : X \to Y$. This is also what allows us to give a sheaf-theoretic interpretation of intersection cohomology.

3.1 General t-structures

Definition 5. *Let $(\mathcal{C}, T, \Delta)$ be a triangulated category, then a t-structure is a pair of full subcategories $\mathbf{C}^{\leq 0}, \mathbf{C}^{\geq 0} \subset \mathcal{C}$ that are closed under isomorphisms, such that the following axioms hold:*

- *The subcategory $\mathcal{C}^{\leq 0}$ is closed under translation by T. In fact we usually write $\mathcal{C}^{\leq -1} = T\mathcal{C}^{\leq 0}$. Similarly we have that $\mathcal{C}^{\geq 0}$ is closed under T^{-1} and we write $\mathcal{C}^{ge1} = T^{-1}\mathcal{C}^{\leq 0}$.*
- *For $Z \in \mathbf{C}^{\leq 0}$ and $Y \in \mathbf{C}^{\geq 0}$ we have*

$$\hom(Z, Y[1]) = 0$$

- *For all objects $X \in \mathbf{C}$ there is a distinguished triangle*

$$Z \to X \to Y \to Z[1]$$

  with $Z \in \mathbf{C}^{\leq 0}$ and $Y \in \mathbf{C}^{ge0}$.

Example 1. If $\mathbf{A}$ is a sufficiently nice abelian category, then the derived category $D^b(\mathbf{A})$ has a t-structure given by complexes with cohomology concentrated in degree ≤ 0 (resp ≥ 0).

Remark 2. Given a t-structure on a triangulated category, we can always shift it to obtain a new t-structure.

Proposition 6. *Define $C^{\heartsuit} = \mathcal{C}^{\leq 0} \cap \mathcal{C}^{\geq 0}$, it is called the heart of the t-structure and is an abelian category.*

Proposition 7. *The inclusion functor $i : \mathcal{C}^{\leq n} \to \mathbf{C}$ has a right adjoint $\tau^{\leq n}$ which we will refer to as a truncation functor. Similarly the inclusion $i : \mathcal{C}^{\geq n}$ has a left adjoint $\tau^{\geq n}$ also called a truncation functor.*

Definition 6. *If $K in \mathcal{C}$ then we define the cohomology functors (associated to the t-structure) to be*

$$H^n(X) = \tau^{\leq 0}\tau^{\geq 0}X[n],$$

this coincides with the usual thing when $\mathcal{C}$ is the derived category of an abelian category with the usual t-structure.

3.2 The perverse t-structure

In this section we let X be a scheme of finite type over a field k. We are going to define the perverse t-structure on $D(X)$, sketch a proof that it is in fact a t-structure and discuss what happens for smooth X.

Definition 7. *We say that K is in ${}^pD(X)^{\leq 0}$ if $\dim \mathrm{Supp}\left(\mathcal{H}^{-i}K\right) \leq i$ for all $i \in \mathbb{Z}$ and that K is in ${}^pD(X)^{\geq 0}$ if DK is in ${}^pD(X)^{\leq 0}$.*

Proposition 8. *This is a t-structure, and we call the heart* $\mathrm{Perv}(X) := \left({}^pD(X)^{\leq 0} \cap {}^pD(X)^{\geq 0}\right)$ *the abelian category of perverse sheaves.*

Note that the perverse t-structure is self dual by definition, i.e., K is a perverse sheaf if and only if DK is a perverse sheaf. In order to make sense of this definition (which is not very intuitive), we will study what the conditions say when X is smooth variety and K a complex of smooth sheaves.

3.3 The smooth case

In this section we will study what happens when X is smooth and equi-dimensional of dimension d. In this case the dualizing complex K_X 'is' of the form $\overline{\mathbb{Q}_l}[2d](d)$, i.e., it is concentrated in degree $-2d$ and there is some Tate twist. If $\mathcal{F}$ is a smooth sheaf on X, then we define $\mathcal{F}^\vee = \mathscr{H}om(\mathcal{F}, \overline{\mathbb{Q}_l})$. We call a complex K a smooth complex if all its cohomology sheaves are smooth sheaves.

Remark 3. There is always a dense open subscheme U of X such that $K\big|_U$ is a smooth scheme. Indeed, every cohomology sheaf $\mathcal{H}^iK$ is constructible and so smooth on a dense open subset U_i and we just define $U = \bigcap_i U_i$.

Proposition 9 (Proposition III.2.1 in [KW01])**.** *For a smooth complex K we have*

$$\mathcal{H}^i(DK) = \mathcal{H}^{-i-2d}(K)^\vee(d).$$

Proof. This is straightforward when $K = \mathcal{F}$ is a smooth sheaf, as the dualizing complex has an explicit description. The general case follows from induction on the degree in which there are non-vanishing cohomology sheaves, see loc. cit. for details. ☐

Corollary 1. *A smooth complex K is a perverse sheaf if and only if $K = \mathcal{F}[d]$ for a smooth sheaf $\mathcal{F}$*

Proof. Note that the cohomology sheaves $\mathcal{H}^iK$ are smooth, and so the dimension of their support is d or 0. The fact that

$$\dim \mathrm{Supp}\, \mathcal{H}^{-i}(K) \leq i$$

then tells us that K has nonzero cohomology sheaves only in degrees $\leq -i$. The dual statement that

$$\dim \mathrm{Supp}\, \mathcal{H}^{-i-2d}(K) = \dim \mathrm{Supp}\, \mathcal{H}^{-i}(K) \leq i$$

tells us that K has nonzero cohomology sheaves only in degrees $\geq -i$. ☐

This is how one shows in practice that the perverse t-structure is actually a t-structure. Roughly speaking we can write $X = U \cup Z$ where U is a dense open smooth subscheme and Z is the closed complement with $\dim Z < \dim X$. Then we know (by induction on the dimension) that we have a t-structure on Z and we can glue it to the t-structure on U defined by shifting the standard one by d. This doesn't quite work as stated, and a detailed proof can be found in Section III.3 of [KW01]. In particular the following Lemma is a crucial ingredient in the proof.

Lemma 1 (Lemma III.3.1 of [KW01]). *If $i : Z \to X$ is a closed immersion with open complement $j : U \to X$ then we have the following results*

$$K \in {}^pD^{\leq 0}(X) \Leftrightarrow j^*K \in {}^pD^{\leq 0}(U) \quad \textit{and } i^*K \in^p D^{\leq 0}(Z)$$
$$K \in {}^pD^{\geq 0}(X) \Leftrightarrow j^!K \in {}^pD^{\geq 0}(U) \quad \textit{and } i^!K \in^p D^{\geq 0}(Z)$$

Proof. The first condition is follows from exactness of j^* and i^*. This exactness means that

$$f^*\mathcal{H}^iK = \mathcal{H}^i f^*$$

where f is j or i. Moreover, the dimension of the support of the sheaf $\mathcal{H}^iK$ is determined by knowing its intersections with U and Z. The second statement follows from Verdier duality, or that

$$f^! \circ D = D \circ f^*$$
$$f^* \circ D = D \circ f^!$$

□

3.4 Functoriality and exactness

Definition 8. *A (triangulated) functor $F : \mathcal{C} \to \mathcal{D}$ between triangulated categories with t-structures is called t-right exact if*

$$F(\mathcal{C}^{\leq 0}) \subset \mathcal{D}^{\leq 0}$$

and t-left exact if

$$F(\mathcal{C}^{\geq 0}) \subset \mathcal{D}^{\geq 0}$$

If it is both t-left and t-right exact then it is called t-exact. This coincides with the usual definition when $\mathcal{C}$ and $\mathcal{D}$ are derived categories of abelian categories.

Lemma 2 (Lemma III.4.1 in [KW01]). *If $i : Z \to X$ is a closed immersion with open complement $j : U \to X$ then we have the following exactness results*

$$\begin{aligned} &j_!, i^* && \textit{are right t-exact} \\ &i_! = i_*, j^! = j^* && \textit{are t-exact} \\ &j_*, i^! && \textit{are left t-exact} \end{aligned}$$

3.5 Intermediate extensions and simple perverse sheaves

Let $j : U \to X$ be an open immersion with closed complement $i : Z \to X$ (everything of finite type over k) as usual. Then we will study perverse sheaves $\overline{K}$ on X that restrict to a fixed perverse sheaf K on U, these will be called extensions of K. The morphism

$$j_! K \to j_* K$$

induces a map

$${}^p H^0(j_! K) \to^p H^0(j_* K)$$

and we will denote its image by $j_{!*} K$.

Proposition 10. *The following are equivalent for $\overline{K} \in \mathrm{Perv}(X)$ extending K.*

- *We have $\overline{K} = j_{!*} K$.*
- *The perverse sheaf $\overline{K}$ has no quotients or subobjects of the form $i_* L$ for $L \in \mathrm{Perv}(Z)$.*
- *We have $i^* \overline{K} \in {}^p D^{\leq -1}(Z)$ and $i^! \overline{K} \in {}^p D^{\geq 1}(Z)$.*

Remark 4. The third equivalent condition is self-dual, and so we find that

$$j_{!*} D K = D j_{!*} K$$

Definition 9. *Let X be a scheme of finite type over k, let U be the smooth locus of X and let $\mathcal{F}$ be a smooth sheaf on U. Then we define the intersection cohomology complex*

$$\mathrm{IC}(X, \mathcal{F}) = j_{!*} \mathcal{F}[d]$$

where d is the dimension of U. If $\mathcal{F}$ is an irreducible local system than this is a simple perverse sheaf.

Theorem 3. *The category of perverse sheaves* $\mathrm{Perv}(X)$*is artinian and noetherian, i.e., all objects are finite successive extensions of simple objects. All simple objects are of the form $K = i_* j_{!*} \mathcal{F}[d]$ for an irreducible closed subscheme $i : Y \to X$, an open dense essentially smooth subscheme $j : U \to Y$ and $\mathcal{F}$ an irreducible smooth sheaf on U.*

Theorem 4. *Let $f : X \to Y$ be a proper morphism of separated finite type schemes over k and let K be a pure perverse sheaf, for example $j_{!*} \mathcal{F}[-d]$ where $j : X^{sm} \to X$ is the inclusion of the smooth locus of X and $\mathcal{F}$ is a pure and smooth sheaf on U. Then*

$$f_* K = \bigoplus_i {}^p\mathcal{H}^i(f_* K)[-i]$$

and moreover it is a semi-simple perverse sheaf.

4 Nearby cycles and Milnor fibers

In this section we give a quick introduction to nearby cycles. We start by giving some intuition from the classical theory of milnor fibers.

4.1 Introduction

If $f : \mathbb{C}^n \supset U \to \mathbb{C}$ is a holomorphic function (say a polynomial) such that $X_0 := f^{-1}(0)$ is a (possibly) singular variety , then the fibers $X_t := f^{-1}(t)$ will be smooth, say for $0 < |t| < \epsilon \ll 1$.

Let Δ_ϵ be the disk of radius ϵ and let Δ_ϵ^* be the corresponding punctured disk. Then we get a fibration

$$f^{-1}\left(\Delta_\epsilon^*\right) \to \Delta_\epsilon^*$$

and we have an action on $H^i(X_t)$ by the fundamental group of Δ_ϵ^* (which is just $\mathbb{Z}$). Note moreover that X_0 is homotopy equivalent to X since Δ_ϵ is contractible, which gives us a specialisation map

$$H^i(X_0) = H^i(f^{-1}\Delta_\epsilon) \to H^i(X_t)$$

whose image lies in the invariant of $H^i(X_t)$ under the monodromy action. These singularities were first studied by Milnor in [Mil68].

We can define the sheaf of nearby cycles by using the following diagram

$$\begin{array}{ccccccc}
X_0 & \xrightarrow{i} & X & \xleftarrow[j]{} & X^* & \xleftarrow[p]{} & \tilde{X}^* \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow \\
\{0\} & \xrightarrow{i} & \Delta_\epsilon & \xleftarrow[j]{} & \Delta_\epsilon^* & \xleftarrow[p]{} & \tilde{\Delta_\epsilon^*},
\end{array}$$

where $\tilde{\Delta_\epsilon^*}$ is the universal cover of Δ_ϵ^* and the rightmost square is cartesian. We can now define

$$R\psi\mathbb{C} = i^* R(j \circ p)_* \mathbb{C},$$

which has an action of $\mathbb{Z}$ because is a $\mathbb{Z}$-covering. The cohomology groups

$$(R^i\psi\mathbb{C})_x$$

compute the i-th cohomology of the Milnor fibre around x. Roughly speaking this is the intersection of a small ball around x (in $\mathbb{C}^n$) with X_0.

4.2 Étale version

A reference for the following section is [Ill]. Let $\Delta = \operatorname{Spec} S$ where S is an Henselian DVR (e.g. $S = \mathbb{Z}_p$) with generic point η and closed point s. Choose a strict henselisation $\tilde{S}$ of S (e.g. $\mathbb{Z}_p^{\mathrm{ur}}$, the ring of integers in the maximal unramified extension) with generic point $\tilde{\eta}$ and closed point $\tilde{s}$. Finally choose a geometric point $\overline{\eta} \to \tilde{S}$ lying over the generic point of $\tilde{S}$. Consider the following diagram induced by base change

$$\begin{array}{ccccc}
X_{\overline{\eta}} & & & & \\
\downarrow & \searrow^{\overline{j}} & & & \\
X_{\tilde{\eta}} & \xrightarrow{\tilde{j}} & X_{\tilde{S}} & \xleftarrow[\tilde{i}]{} & X_{\tilde{s}} \\
\downarrow & & \downarrow & & \downarrow \\
X_{\eta} & \xrightarrow{j} & X & \xleftarrow[i]{} & X_s,
\end{array}$$

and define

$$\begin{aligned} R\Psi : D(X_{\overline{\eta}}) &\to D(X_{\tilde{s}}) \\ K &\mapsto \tilde{i}^* R\overline{j}_* K, \end{aligned}$$

the complex of nearby cycles associated to K.

Theorem 5. *If K is a perverse sheaf on $X_{\overline{\eta}}$, then so is $R\psi K$.*

Proof. It is a deep result of Gabber that nearby cycles commute with duality, i.e.,

$$DR\Psi K = R\Psi DK.$$

Therefore it suffices to prove that $R\Phi$ is left exact for the perverse t-structure. The following Lemma shows that j_* is t-exact and we already know that i^* is left exact for the t-structure so we are done. □

Lemma 3. *If $j : U \to X$ is an affine open immersion (like the open complement of the inclusion of a divisor) then $j_* = Rj_*$ is exact for the perverse t-structure.*

Proof. The t-left exactness of j_* follows from Lemma 2. For the t-right exactness, the main ingredient is (relative) Artin vanishing for affine schemes ([Gro73])

Theorem 6 (Théorème 3.1 (d) SGA4 Exposé XIV)**.** *For an affine morphism $f : X \to Y$ and $\mathcal{F}$ a constructible sheaf supported in dimension $\leq d$ we have*

$$\dim \operatorname{Supp} R^q f_* \leq d - q.$$

Now the result follows by applying the following hypercohomology spectral sequence

$$E_2^{p,q} = R^p j_* \mathcal{H}^q(K) \Rightarrow \mathcal{H}^{p+q}(Rj_* K)$$

and noting that $\mathcal{H}^q(K)$ is supported in dimension $\leq -q$ and so $R^p j_* \mathcal{H}^q(K)$ is supported in dimension $\leq -q - p$ by the theorem, precisely the required statement. □

Theorem 7. *Assume that $f : X \to \Delta$ is proper or that X is a nice integral model of a Shimura variety, then*

$$\mathbb{H}^i(X_{\overline{\eta}}, \overline{\mathcal{Q}_l}) = \mathbb{H}^i(X_{\overline{s}}, R\Psi \mathbb{Q}_l).$$

Here we can replace the coefficient sheaf $\overline{\mathbb{Q}_l}$ with more general automorphic local systems [LS18]. In the proper case this result holds with any choice of coefficients by proper base change

References

[Mil68] John Milnor. *Singular points of complex hypersurfaces.* Annals of Mathematics Studies, No. 61. 1968, pp. iii+122.

[Gro73] Alexander Grothendieck. *Théorie des topos et cohomologie étale des schémas. Tome 3.* Lecture Notes in Mathematics, Vol. 305. Springer-Verlag, Berlin-New York, 1973, pp. vi+640. URL: http://fabrice.orgogozo.perso.math.cnrs.fr/SGA4/index.html.

[Del77] P. Deligne. *Cohomologie étale.* Vol. 569. Lecture Notes in Mathematics. 1977, pp. iv+312.

[Del80] Pierre Deligne. "La conjecture de Weil. II". In: *Inst. Hautes Études Sci. Publ. Math.* 52 (1980), pp. 137–252. ISSN: 0073-8301. URL: http://www.numdam.org/item?id=PMIHES_1980__52__137_0.

[GM80] Mark Goresky and Robert MacPherson. "Intersection homology theory". In: *Topology* 19.2 (1980), pp. 135–162. ISSN: 0040-9383. DOI: 10.1016/0040-9383(80)90003-8. URL: https://doi.org/10.1016/0040-9383(80)90003-8.

[BBD82] A. A. Beılinson, J. Bernstein, and P. Deligne. "Faisceaux pervers". In: *Analysis and topology on singular spaces, I (Luminy, 1981).* Vol. 100. Astérisque. Soc. Math. France, Paris, 1982, pp. 5–171.

[KW01] Reinhardt Kiehl and Rainer Weissauer. *Weil conjectures, perverse sheaves and l'adic Fourier transform.* Vol. 42. Springer-Verlag, Berlin, 2001, pp. xii+375. ISBN: 3-540-41457-6. DOI: 10.1007/978-3-662-04576-3. URL: https://doi.org/10.1007/978-3-662-04576-3.

[Con07] Brian Conrad. "Deligne's notes on Nagata compactifications". In: *J. Ramanujan Math. Soc.* 22.3 (2007), pp. 205–257. ISSN: 0970-1249.

[BS15] Bhargav Bhatt and Peter Scholze. "The pro-étale topology for schemes". In: *Astérisque* 369 (2015), pp. 99–201. ISSN: 0303-1179.

[Lur17] Jacob Lurie. *Higher Algebra.* 2017. URL: http://www.math.harvard.edu/~lurie/papers/HA.pdf.

[LS18] Kai-Wen Lan and Benoît Stroh. "Nearby cycles of automorphic étale sheaves". In: *Compos. Math.* 154.1 (2018), pp. 80–119. ISSN: 0010-437X. DOI: 10.1112/S0010437X1700745X. URL: https://doi.org/10.1112/S0010437X1700745X.

[Sta18] The Stacks project authors. *The Stacks project.* 2018.

[Ill] L. Illusie. *Grothendieck and vanishing cycles.* URL: https://www.math.u-psud.fr/~illusie/Illusie-Grothendieck-nearby-cycles5.pdf.